

Binary Relation

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Adapted From:

Stanford CS103 Binary Relation

《数理逻辑与集合论》9.3.4, 10.1-10.4, 10.5.1, 10.6, 10.7

Review

- **ZF Axiom System**
 - Russell's paradox in naïve set theory.
 - Subset axiom schema, Axiom of regularity
- **Cardinality of Sets**
 - Cardinality of finite sets. Principle of inclusion and exclusion.
 - Cardinality of infinite sets. Comparing cardinality.
 - Cantor's Theorem.
 - $|Problems| > |Programs|$

How we investigate computability?

2.1 Discussion on Problems

Part1. Intro & Set theory(I) : Basics & Formal Language.

Part2. Set Theory(II): Axiom system & Cardinality.

Part3. Capture Structures: Binary Relation & Function

2.2 Discussion on Computation

Part1. Turing Machine Basics.

Part2. Variants of Turing Machine. Church-Turing Thesis.

2.3 Discussion on computability

Part1. The Language of Turing Machine. R & RE

Part2. Undecidability.

What is the structure of a problem?

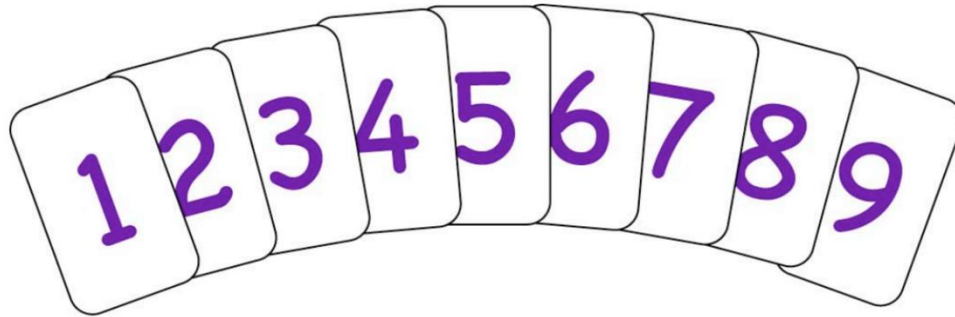
Tic Tac Toe

- Two players take turns marking the spaces in a 3×3 grid.
- The player who succeeds in placing three of their marks in a horizontal, vertical, or diagonal row is the winner.

X	X	O
X	O	O
O	X	

15 Points

- Two players take turns to select a card with points from 1 to 9.
- The player who picks three cards whose sum equals 15 wins.



Take a deeper look.

X	X	O
X	O	O
O	X	

8	1	6
3	5	7
4	9	2

*They only differ in their surface structure, but
are based on an identical deep structure!!*

Structure of a Problems

- A collection of objects.
 - Cells in Tic Tac Toe.
 - Cards in 15 points.
 - Students in cdm class.
- The connection between objects.
 - Three cells are in a same horizontal, vertical, or diagonal row.
 - Sum of three cards equals 15.
 - Two students come from the same city.

Structure of a Problems

- A collection of objects.
 - Cells in Tic Tac Toe.
 - Cards in 15 points.
 - Students in cdm class.
- The connection between objects.
 - Three cells are in a same horizontal, vertical, or diagonal row.
 - Sum of three cards equals 15.
 - Two students are in same desk.

Set Theory

Structure of a Problems

- A collection of objects.
 - Cells in Tic Tac Toe.
 - Cards in 15 points.
 - Students in cdm class.
- The connection between objects.
 - Three cells are in a same horizontal, vertical, or diagonal row.
 - Sum of three cards equals 15.
 - Two students are in same desk.

Relation & Function

What are relations?

Relations

- We've seen a lot of relations:

- Between sets:

$$A = B, A \subseteq B, A \neq B$$

- Between numbers:

$$x < y, x = y, x \equiv_k y$$

$x \equiv_k y$ means $x \% k = y \% k$

- Between cards:

$$c_1 + c_2 + c_3 = 15$$

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Connection between two objects
Binary Relation

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*Connection between three objects
Trinary Relation*

Relations

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- Between sets:

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- Between numbers:

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- Between cards:

$$c_1 + c_2 + c_3 = 15$$

- **We only focus on binary relations.**

Ordered Pair

Ordered pair $\langle a, b \rangle$ is collection of two elements a, b that has a as its first element and b as its second elements.

- Ordered pair $\langle a, b \rangle$ satisfies:
 - $a \neq b \Rightarrow \langle a, b \rangle \neq \langle b, a \rangle$
 - $\langle a, b \rangle = \langle c, d \rangle \Leftrightarrow a = c \wedge b = d$
- Examples:
 - $\langle 1, 2 \rangle$ and $\langle 2, 1 \rangle$ are two different ordered pair.
 - $\langle 1, 1 \rangle$ is also an ordered pair.

Ordered Pair and Set

- We can use set to describe an ordered pair.
- Define that : $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$
- If $a = b$:

$$\langle a, b \rangle = \{\{a\}\}$$

- If $a \neq b$:
$$\langle a, b \rangle = \{\{a\}, \{a, b\}\} \neq \{\{b\}, \{b, a\}\} = \langle b, a \rangle$$

- If $\langle a, b \rangle = \langle c, d \rangle$:
$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} \Leftrightarrow a = c \wedge b = d$$

Cartesian Products

The Cartesian product of the sets A, B , denoted by $A \times B$, is the set of ordered pair $\langle a, b \rangle$ where $a \in A$ and $b \in B$

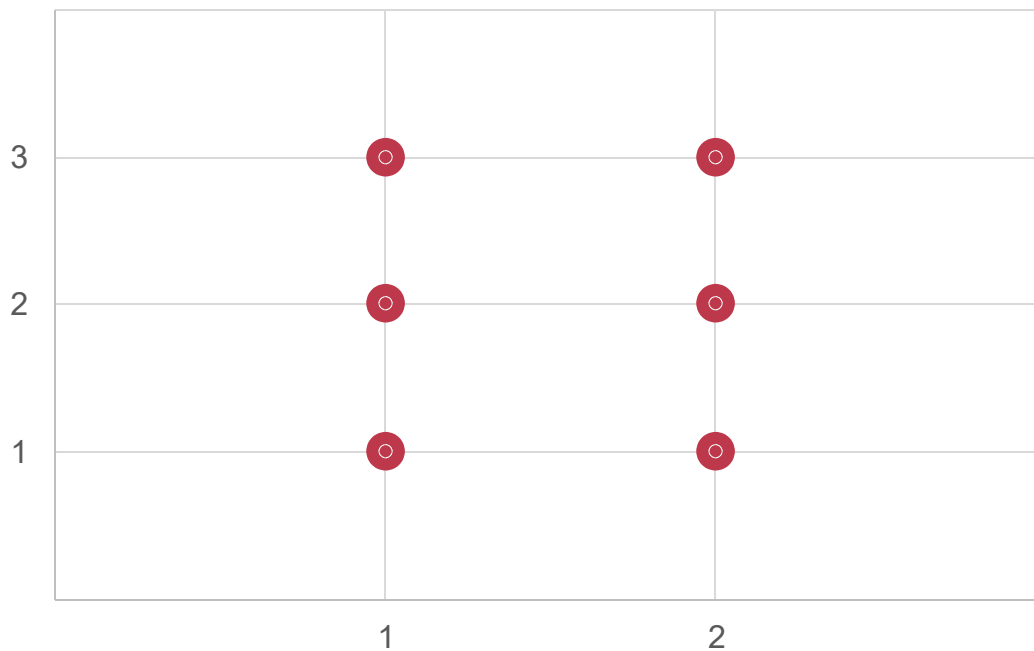
$$A \times B = \{\langle a, b \rangle | a \in A \wedge b \in B\}$$

- Let $A = \{1,2\}, B = \{1,2,3\}$
 - $\emptyset \times A = B \times \emptyset = \emptyset$
 - $A \times B = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 2,3 \rangle\}$
 - $B \times A = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle, \langle 3,2 \rangle\} \neq A \times B$

Cartesian Products

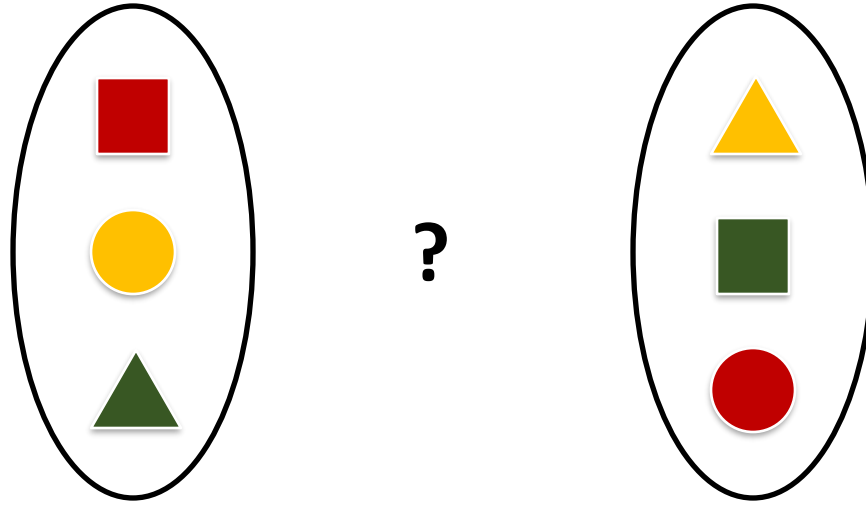
$$A \times B$$

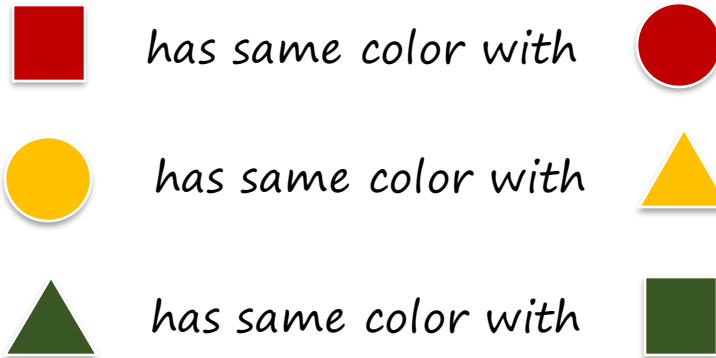
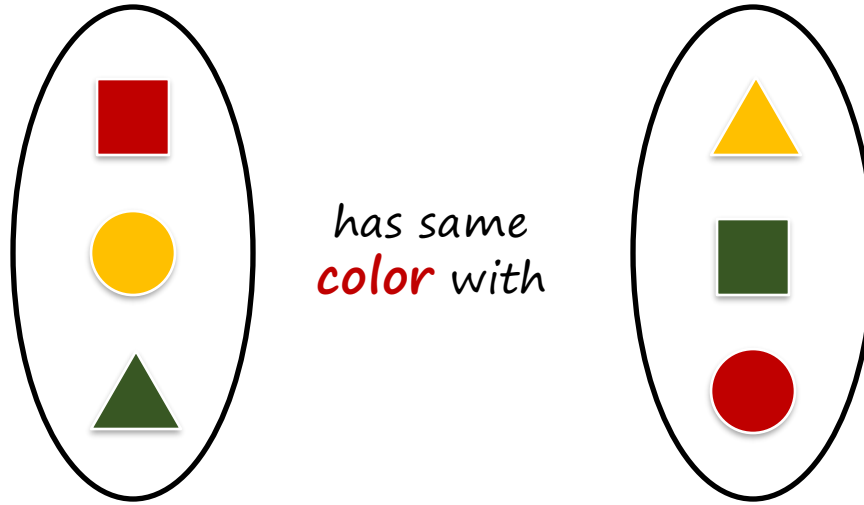
$$B = \{1,2,3\}$$

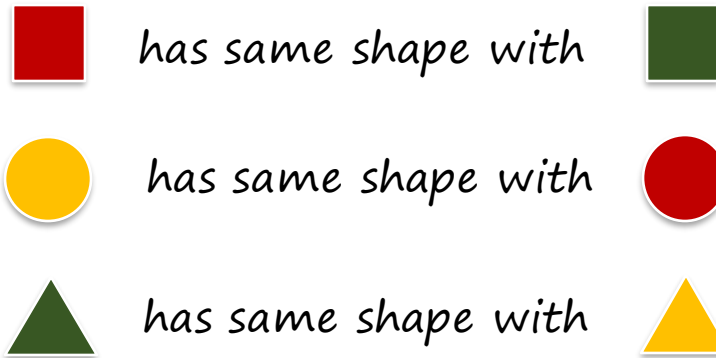
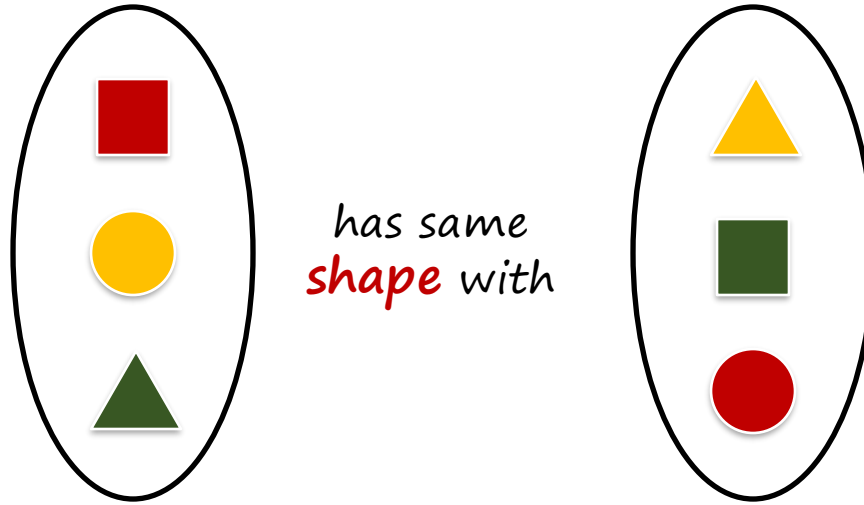


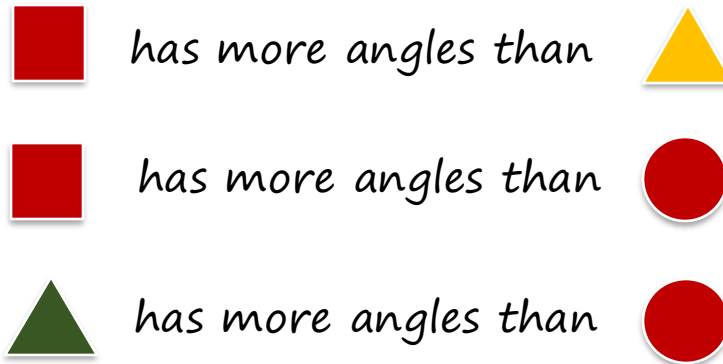
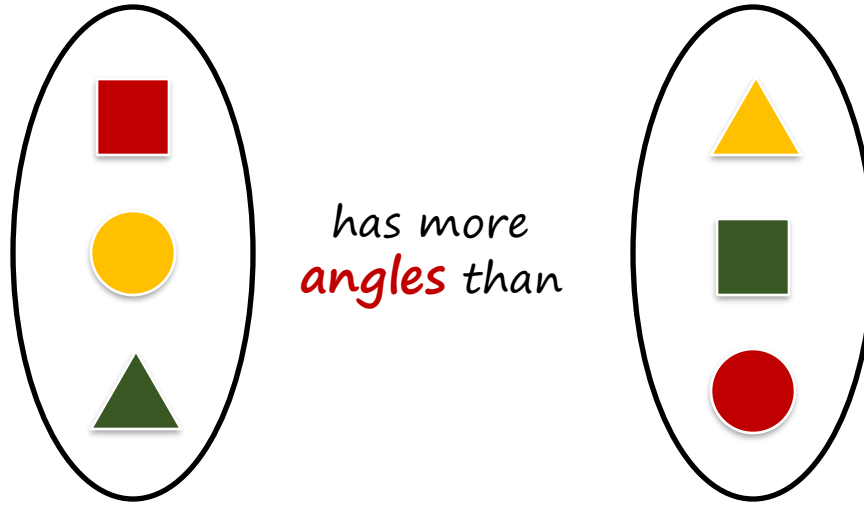
$$A = \{1,2\}$$

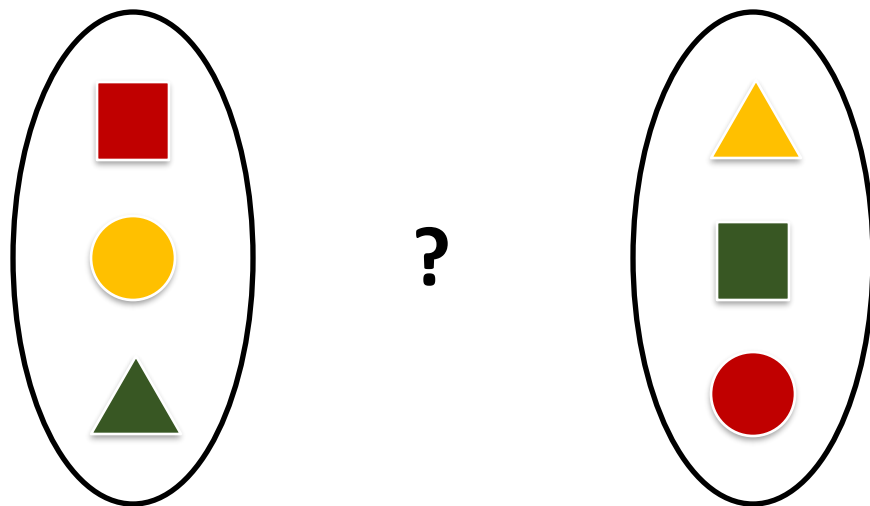
What exactly is a binary relation?











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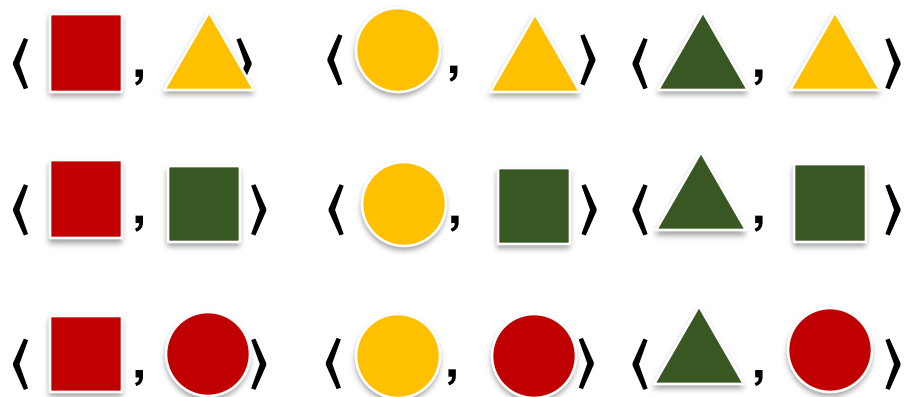
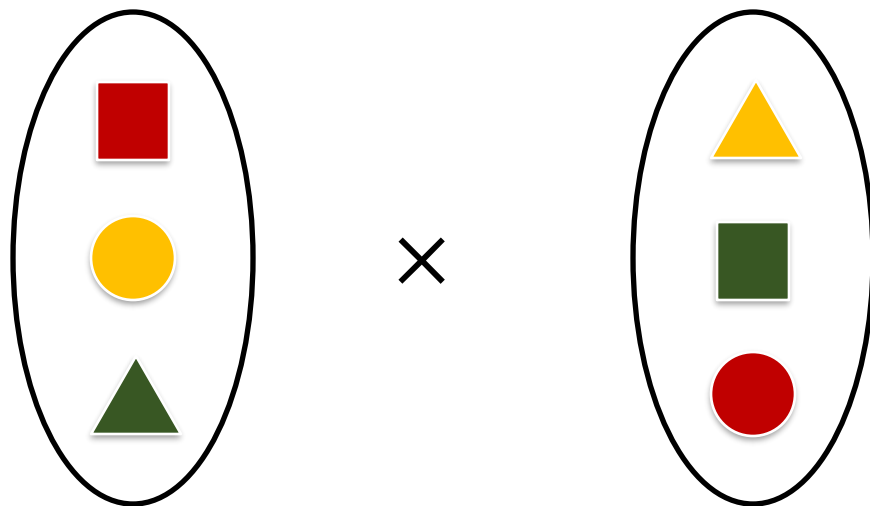


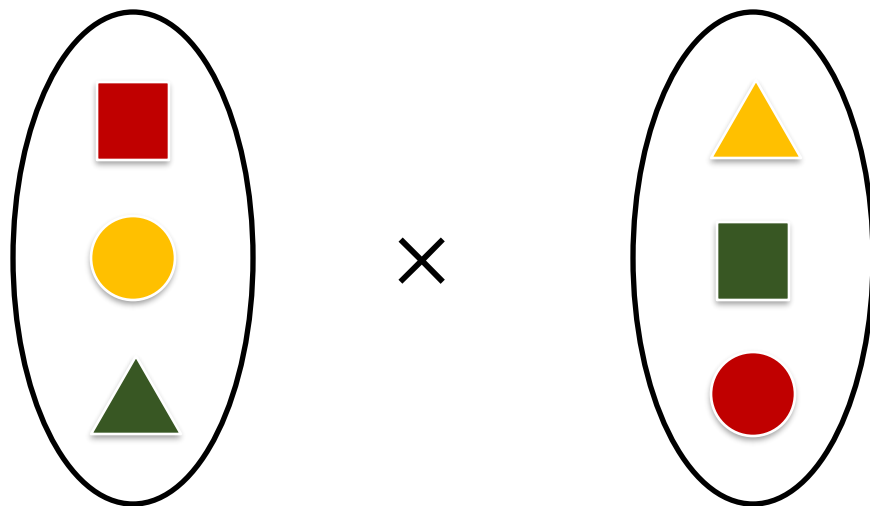
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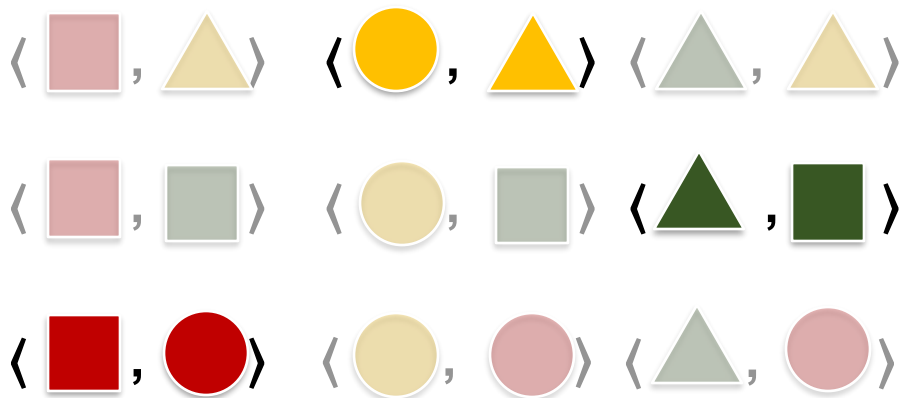
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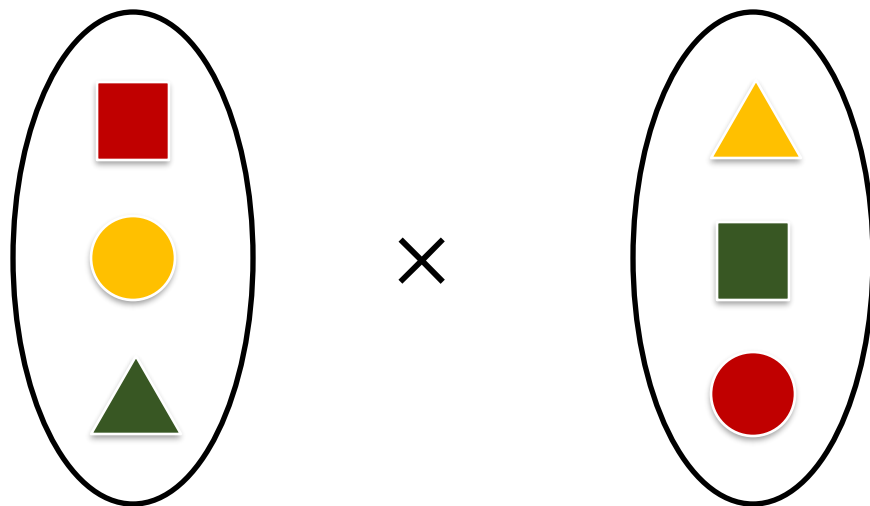






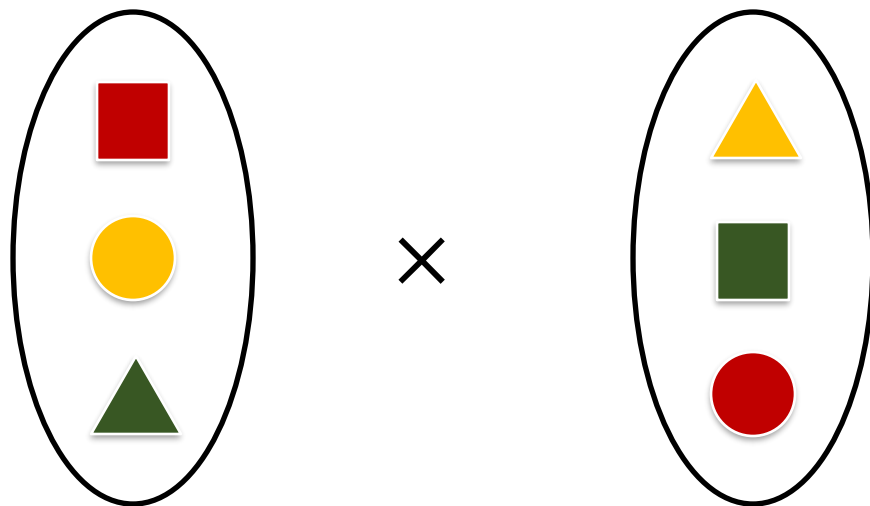
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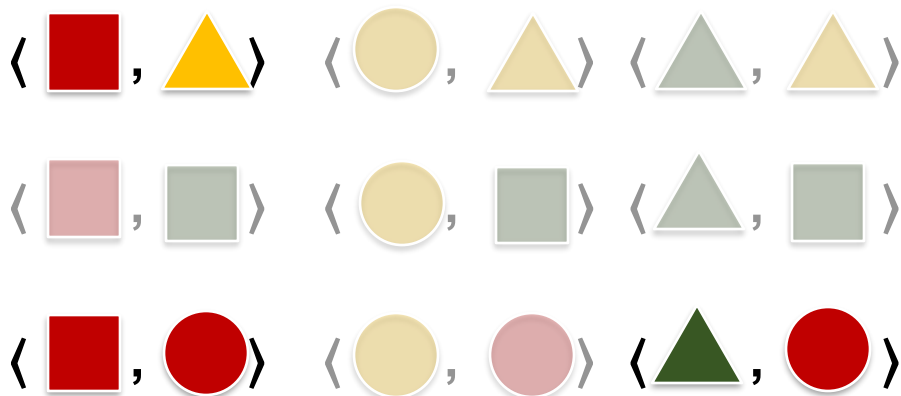


has same *shape* with





has more *angles* than



Binary Relation

Let A, B be sets. A binary relation from A to B is a subset of $A \times B$.

- In other words, a binary relation from A to B is a set R of **ordered pairs**, where the first element of each ordered pair comes from A and the second element comes from B .
 - aRb means $\langle a, b \rangle \in R$
 - $a \not R b$ means $\langle a, b \rangle \notin R$
- A relation on a set A is a relation from A to A , or is a subset of $A \times A$.

Domain, Range and Field

- For a relation ***R*** from ***A*** to ***B***, we define:

- The domain(定义域)of *R* is:

$$dom(R) = \{x | (\exists y)(\langle x, y \rangle \in R)\}$$

- The range(值域)of *R* is:

$$ran(R) = \{y | (\exists x)(\langle x, y \rangle \in R)\}$$

- The field(域)of *R* is:

$$fld(R) = dom(R) \cup ran(R)$$

Binary Relation

- For set $A = \{1,2,3\}$, define:
 - $R_1 = \{\langle x, y \rangle | x \in A \wedge y \in A \wedge x|y\}$
 - $R_2 = \{\langle x, y \rangle | x \in A \wedge y \in A \wedge x < y\}$
- Then
 - $1R_12, 2R_23$
 - $R_1 = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle\}$
 - $R_2 = \{\langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,3 \rangle\}$
 - $dom(R_1) = \{1,2,3\}, ran(R_1) = \{1,2,3\}, fld(R_1) = \{1,2,3\}$
 - $dom(R_2) = \{1,2\}, ran(R_2) = \{2,3\}, fld(R_2) = \{1,2,3\}$

Special Relations

- For any set A , there are three kinds of special relations.

- The identity relation (恒等关系) :

$$I_A = \{\langle x, x \rangle | x \in A\}$$

- The universal relation (全域关系) :

$$E_A = \{\langle x, y \rangle | x \in A \wedge y \in A\}$$

- The empty relation (空关系) :

$$N_A = \emptyset$$

- For arbitrary elements $x \in A$ and $y \in A$:

- xN_Ay holds never
- xE_Ay holds always
- xI_Ay holds if and only if $x = y$

Relation Matrix(关系矩阵)

- Suppose that R is a relation from $A = \{a_1, a_2, a_3, \dots, a_m\}$ to $B = \{b_1, b_2, b_3, \dots, b_n\}$. Then the relation R can be represented by zero-one matrix $M_R = (m_{ij})_{m \times n}$, where:

$$m_{ij} = \begin{cases} 1 & \text{if } \langle a_i, b_j \rangle \in R \\ 0 & \text{if } \langle a_i, b_j \rangle \notin R \end{cases}$$

- If R is a relation on $A = \{a_1, a_2, a_3, \dots, a_m\}$, then the relation matrix of R is a $m \times m$ square $M_R = (m_{ij})_{m \times m}$

Relation Matrix: Example

Let $A = \{1, 2, 3, 4\}$, the relation R_1 on A is:

$$R_1 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle, \langle 2, 4 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 1 \rangle\}$$

The relation matrix of R_1 is:

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Relation Matrix: Example

Identity relation I :

$$M_I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Relation Matrix: Example

Universal relation E :

$$M_E = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

Relation Matrix: Example

Empty relation N :

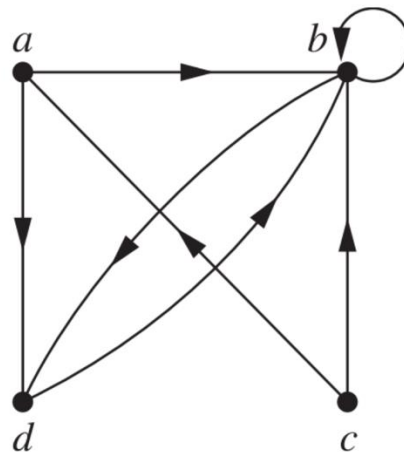
$$M_N = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Digraphs(关系图)

Defination

A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b) , and the vertex b is called the terminal vertex of this edge.

The directed graph with vertices a, b, c , and d , and edges $\langle a, b \rangle, \langle a, d \rangle, \langle b, b \rangle, \langle b, d \rangle, \langle c, a \rangle, \langle c, b \rangle$, and $\langle d, b \rangle$



Digraphs

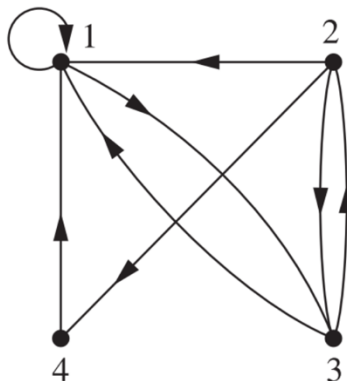
Digraph of Relation R on set A :

- Vertices : Elements of A
- Edges: Ordered pairs in R

The directed graph of the relation

$$R_1 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle, \langle 2, 4 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 1 \rangle\}$$

on the set $\{1, 2, 3, 4\}$ is:



Combining Relations

- Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Let $A = \{1, 2, 3\}$. The relations $R_1 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 3, 2 \rangle\}$ and $R_2 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle\}$ can be combined to obtain:

- $R_1 \cup R_2 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$
- $R_1 \cap R_2 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle\}$
- $R_1 - R_2 = \{\langle 2, 1 \rangle, \langle 3, 2 \rangle\}$
- $R_2 - R_1 = \{\langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle\}$
- $R_1 \oplus R_2 = \{\langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$

Combining Relations: Relation Matrix

Or represent in matrix:

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{then:}$$

$$\begin{aligned} M_{R_1 \cup R_2} &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, & M_{R_1 \cap R_2} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ M_{R_1 - R_2} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & M_{R_2 - R_1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ M_{R_2 \oplus R_1} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{aligned}$$

Inverse of Relations (关系的逆)

Defination

Let R be a relation from a set A to a set B . The inverse relation of R , denoted by R^{-1} , is the set of ordered pairs $\{\langle b, a \rangle \mid \langle a, b \rangle \in R\}$ from B to A

Let $A = \{a, b, c\}$, a relation on A , $R = \{\langle a, b \rangle, \langle a, c \rangle, \langle b, a \rangle, \langle c, b \rangle\}$, then:

$$R^{-1} = \{\langle b, a \rangle, \langle c, a \rangle, \langle a, b \rangle, \langle b, c \rangle\}$$

Inverse of Relations: Relation Matrix

R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$, then :

$$M_{R^{-1}} = M_R^T, \text{ where } M_R^T \text{ is the transpose of } M_R$$

Inverse of Relations: Relation Matrix

R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$, then :

$$M_{R^{-1}} = M_R^T, \text{ where } M_R^T \text{ is the transpose of } M_R$$

Proof:

Assume that $M_R = \langle s_{ij} \rangle_{m \times n}$ where s_{ij} is the element. And $M_{R^{-1}} = \langle t_{ij} \rangle_{n \times m}$, where t_{ij} is the element.

If $s_{ij} = 1$, then we know $\langle a_i, b_j \rangle \in R$, according to the definition of inverse, we have $\langle b_j, a_i \rangle \in R^{-1}$, so $t_{ji} = 1$

If $s_{ij} = 0$, then we know $\langle a_i, b_j \rangle \notin R$. So $\langle b_j, a_i \rangle \notin R^{-1}$, otherwise $\langle a_i, b_j \rangle \in R$, then $t_{ji} = 0$

In any case, we have $s_{ij} = t_{ji}$, and it means M_R and $M_{R^{-1}}$ are transposed each other. Q.E.D

Composite of Relations (关系的合成)

Definition

Let R be a relation from A to B , and S be a relation from B to C . The composite of R and S , denote as $S \circ R$, is:

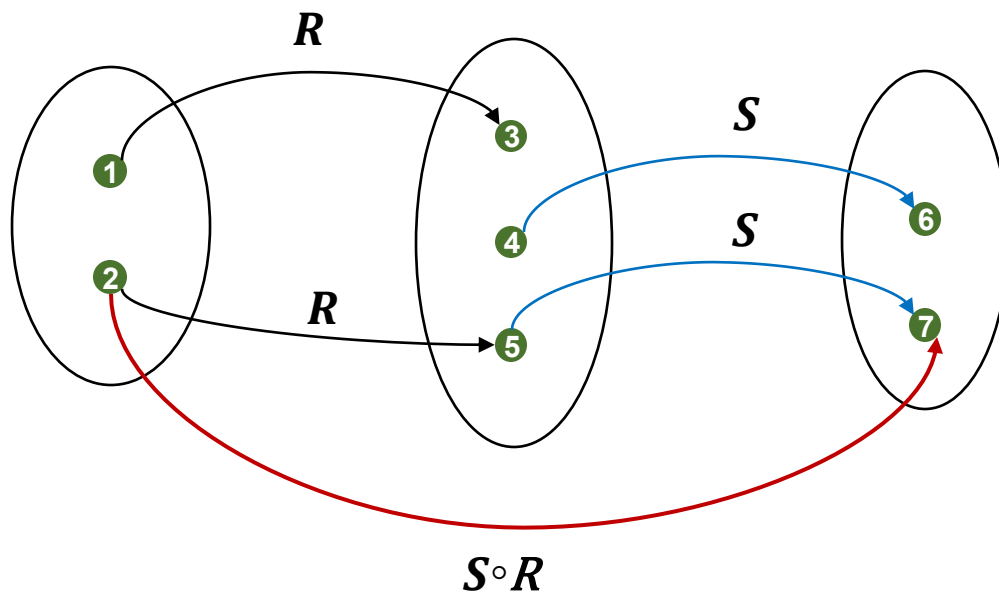
$$S \circ R = \{ \langle x, y \rangle \mid (\exists z)(\langle x, z \rangle \in R \wedge \langle z, y \rangle \in S) \}$$

Let $A = \{1, 2, 3\}$, two relations on A ,

$R = \{ \langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle \}$, $S = \{ \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle, \langle 3, 3 \rangle \}$

Then $S \circ R = \{ \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle \}$

Composite of Relations



Composite of Relations: Relation Matrix

Let A be a set and R and S are two relations on A , the relation matrix of R and S are:

$$M_R = (r_{ij}), M_S = (s_{ij})$$

Then we have:

$$M_{S \circ R} = M_R \odot M_S = (w_{ij})$$

where $\odot$ is the matrix boolean product and $w_{ij} = \bigvee_{k=1}^n (r_{ik} \wedge s_{kj})$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$

$$\begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{1} \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} \boxed{1} & 1 \\ \boxed{0} & 0 \\ \boxed{0} & 1 \end{bmatrix} = \begin{bmatrix} \boxed{(1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0)} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

The diagram illustrates the element-wise multiplication (odot) of two matrices. The first matrix is a 2x3 matrix with rows $[1, 0, 1]$ and $[0, 1, 1]$. The second matrix is a 3x2 matrix with columns $[1, 0, 0]$ and $[1, 0, 1]$. The result is a 2x2 matrix with the top-left element 1. Arrows indicate the dot product of the first row of the first matrix and the first column of the second matrix.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 1) \end{bmatrix}$$

The diagram illustrates the element-wise AND operation between a 2x3 matrix and a 3x1 vector. The first row of the matrix is [1, 0, 1] and the second row is [0, 1, 1]. The vector is [1, 0, 0]. The result is a 1x1 matrix containing the value 1. The calculation is shown as (1 AND 1) OR (0 AND 0) OR (1 AND 1).

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} & 1 & \\ & & 1 \end{bmatrix}$$

Diagram illustrating a binary operation (odot) between two matrices. The first matrix is 2x3 with elements [1, 0, 1] in the first row and [0, 1, 1] in the second row. The second matrix is 3x2 with elements [1, 1] in the first column and [0, 0] in the second column, and [0, 1] in the third column. The result is a 2x2 matrix with elements [1, 1] in the first row and [1, 1] in the second row. Arrows indicate the mapping of elements: the first row of the first matrix maps to the first row of the result, and the second row of the first matrix maps to the second row of the result.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Property of Inverse and Composite

- ① $\text{dom}(R^{-1}) = \text{ran}(R)$
- ② $\text{ran}(R^{-1}) = \text{dom}(R)$
- ③ $(R^{-1})^{-1} = R$
- ④ $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$
- ⑤ $(R \circ S) \circ Q = R \circ (S \circ Q)$
- ⑥ $R_1 \circ (R_2 \cup R_3) = R_1 \circ R_2 \cup R_1 \circ R_3$
- ⑦ $(R_1 \cup R_2) \circ R_3 = R_1 \circ R_3 \cup R_2 \circ R_3$
- ⑧ $R_1 \circ (R_2 \cap R_3) \subseteq R_1 \circ R_2 \cap R_1 \circ R_3$
- ⑨ $(R_1 \cap R_2) \circ R_3 \subseteq R_1 \circ R_3 \cap R_2 \circ R_3$


$$R_1 \circ (R_2 \cap R_3) \subseteq R_1 \circ R_2 \cap R_1 \circ R_3$$

Proof:

$$\langle x, y \rangle \in R_1 \circ (R_2 \cap R_3)$$

$$\Leftrightarrow (\exists z)(\langle x, z \rangle \in (R_2 \cap R_3) \wedge \langle z, y \rangle \in R_1)$$

$$\Leftrightarrow (\exists z)(\langle x, z \rangle \in R_2 \wedge \langle x, z \rangle \in R_3 \wedge \langle z, y \rangle \in R_1)$$

$$\Rightarrow (\exists z)(\langle x, z \rangle \in R_2 \wedge \langle z, y \rangle \in R_1) \wedge (\exists z)(\langle x, z \rangle \in R_3 \wedge \langle z, y \rangle \in R_1)$$

$$\Leftrightarrow \langle x, y \rangle \in (R_1 \circ R_2) \wedge \langle x, y \rangle \in (R_1 \circ R_3)$$

$$\Leftrightarrow \langle x, y \rangle \in (R_1 \circ R_2) \cap (R_1 \circ R_3)$$

Power of Relations (关系的幂)

- For a relation R on set A , $n \in \mathbb{N}$, the n th power of R , denoted by R^n , is :
 - $R^0 = I_A$
 - $R^{n+1} = R^n \circ R$
- $R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$
 - $R^0 = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle\}$
 - $R^1 = R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$
 - $R^2 = R^4 = \{\langle a, a \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle b, d \rangle\}$
 - $R^3 = R^5 = \{\langle a, b \rangle, \langle a, d \rangle, \langle b, a \rangle, \langle b, c \rangle\}$

Power of Relations

- For a relation R on set A , $n \in \mathbb{N}$, the n th power of R , denoted by R^n , is :
 - $R^0 = I_A$
 - $R^{n+1} = R^n \circ R$
- $R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$
 - $R^0 = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle\}$
 - $R^1 = R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$
 - $R^2 = R^4 = \{\langle a, a \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle b, d \rangle\}$
 - $R^3 = R^5 = \{\langle a, b \rangle, \langle a, d \rangle, \langle b, a \rangle, \langle b, c \rangle\}$

Some different power of R are equal.

Power of Relations

A is a finite set and $|A| = n$, R is a relation on A , then there exists natural number $s, t, s \neq t$, such that $R^s = R^t$

Proof:

Any relation on A is a subset of $A \times A$, or an element of $\mathcal{P}(A \times A)$.

We know that $|\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^{|A|^2} = 2^{n^2}$, so there are finite many relations on a finite set A . But there are infinite many powers of R . It means there are different powers of R which are equal, or

$$(\exists s)(\exists t)(s \in \mathbb{N} \wedge t \in \mathbb{N} \wedge s \neq t \wedge R^s = R^t)$$

Q.E.D

Power of Relations

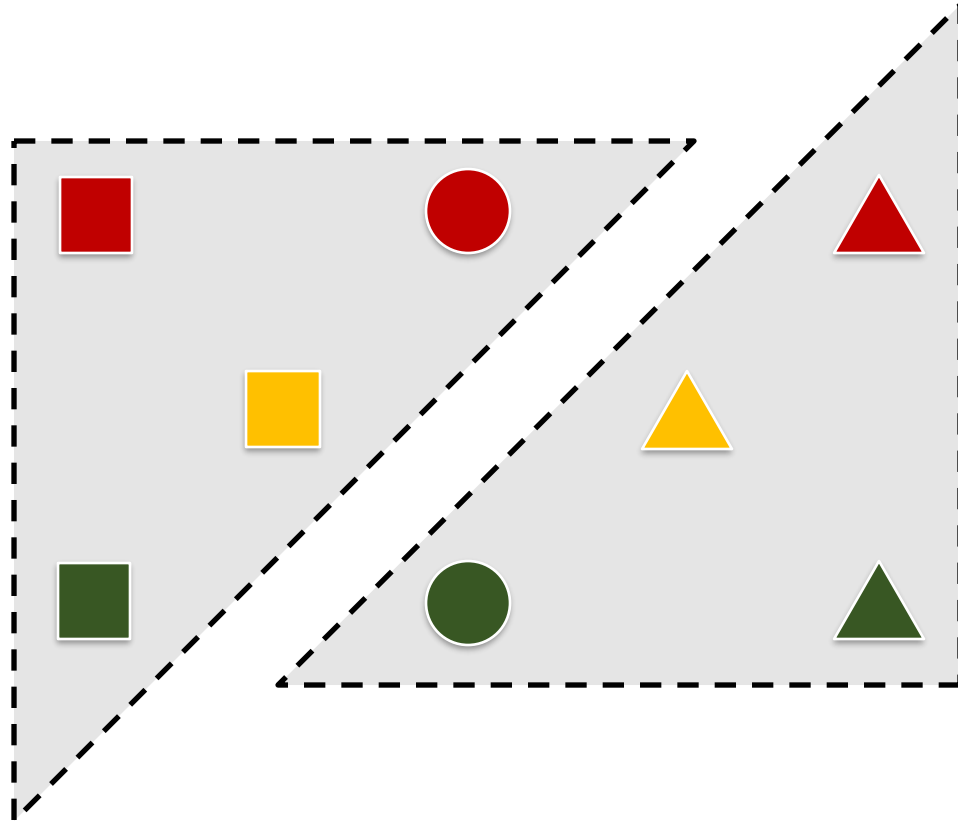
- $R^m \circ R^n = R^{m+n}$
- $(R^m)^n = R^{mn}$
- A is a finite set and $|A| = n$, R is a relation on A , and there exists natural number $s, t, s \neq t$, such that $R^s = R^t$
 - $R^{s+k} = R^{t+k}, k \in \mathbb{N}$
 - $R^{s+kp+i} = R^{t+kp+i}, k \in \mathbb{N}, i \in \mathbb{N}, p = t - s$
 - Let $B = \{R^0, R^1, \dots, R^{t-1}\}$, then for any natural number $q, R^q \in B$

How can we capture a structure with relations?

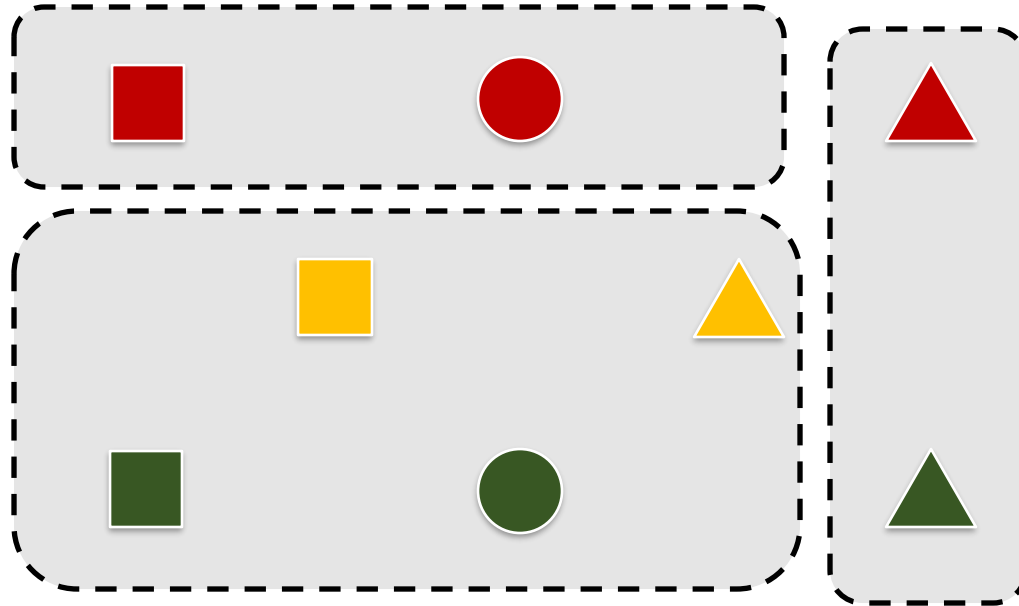
Partitions(划分)



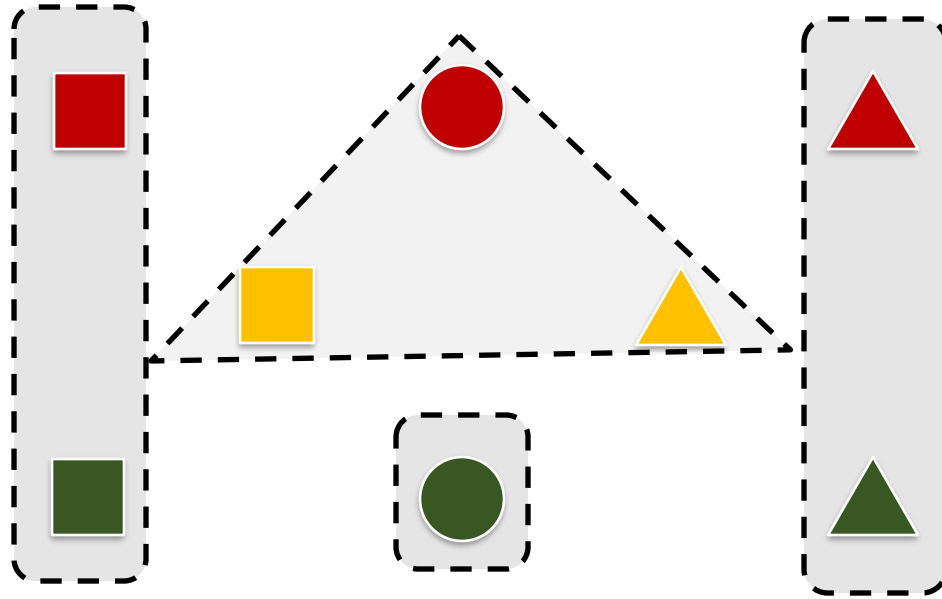
Partitions



Partitions



Partitions



Partitions

- A **partition of a set** is a way of splitting the set into **disjoint, nonempty subsets** so that every element belongs to exactly one subset.
- Intuitively, a partition of a set breaks the set apart into smaller pieces.

Partitions

- More formally, for a non-empty set A , a set π is called the partition of A if:
 - $\emptyset \notin \pi$
 - $(\forall x)(x \in \pi \rightarrow x \subseteq A)$
 - $\bigcup \pi = A$
 - $(\forall x)(\forall y)((x \in \pi \wedge y \in \pi \wedge x \neq y) \rightarrow x \cap y = \emptyset)$

Partitions

- For set $A = \{a, b, c, d\}$
 - $\pi_1 = \{\{a\}, \{b, c\}, \{d\}\}$
 - $\pi_2 = \{\{a, b, c, d\}\}$
 - $\pi_3 = \{\{a, b\}, \{c\}, \{a, d\}\}$
 - $\pi_4 = \{\{a, b, d\}\}$
- } valid partitions
- } not partition

What is the relationship between partitions and relations?

Relation in Partition

- For a partition π of set A , we can define a relation R such that aRb if a and b are in a same partition block.

$$R = \{\langle a, b \rangle | (\exists \pi_0)(\pi_0 \in \pi \wedge a \in \pi_0 \wedge b \in \pi_0)\}$$

- Then we have:
 - aRa for $\forall a \in A$
 - If aRb , then bRa
 - If aRb and bRc , then aRc

Relation in Partition

- For a partition π of set A , we can define a relation R such that aRb if a and b are in a same partition block.

$$R = \{\langle a, b \rangle | (\exists \pi_0)(\pi_0 \in \pi \wedge a \in \pi_0 \wedge b \in \pi_0)\}$$

- Then we have:

- aRa for $\forall a \in A$

An element is in the same partition block with itself.

- If aRb , then bRa

If a and b are in the same block, then b and a are in the same block too.

- If aRb and bRc , then aRc

If a and b are in the same block, b and c are also in the same block, then a and c are in the same block which contains b .

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

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Reflexivity(自反性)

- Some relations always hold from any element to itself.
- Examples:
 - $x = x$ for any x
 - $A \subseteq A$ for any set A
 - $x \geq x$ for any x
- Relations of this sort are called reflexive.

R on A is reflexive $\Leftrightarrow (\forall a)(a \in A \rightarrow aRa)$

Reflexivity: Example

- I_A and E_A
- $=, \geq, \leq$ on $\mathbb{R}$
- $R = \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle\}$ on set $A = \{1,2,3\}$

Reflexivity: Example

- I_A and E_A 😊
 - $aI_A a$ and $aE_A a$ for all $a \in A$
- $=, \geq, \leq$ on $\mathbb{R}$ 😊
 - $a = a, a \geq a, a \leq a$ for all $a \in \mathbb{R}$
- $R = \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle\}$ on set $A = \{1,2,3\}$ 😞
 - $\langle 3,3 \rangle \notin R$, it's not reflexive

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Symmetry(对称性)

- In some relations, the relative order of the objects doesn't matter.
- Examples:
 - If $x = y$ then $y = x$
 - If $x \equiv_k y$ then $y \equiv_k x$
- These relations are called symmetric.

$$\begin{aligned} & \textbf{R on A is symmetric} \Leftrightarrow \\ & (\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa) \end{aligned}$$

Symmetry : Example

- I_A and E_A
- N_A
- $R = \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$

Symmetry : Example

- I_A and E_A 😊
 - $aI_Ab \rightarrow bI_Aa$, $aE_Ab \rightarrow bE_Aa$
- N_A 😊
 - $aN_Ab \rightarrow bN_Aa$ always hold cause aN_Ab never holds
- $R = \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$ 😊

Symmetry : Relation Matrix

R is symmetric $\Leftrightarrow M_R$ is symmetric matrix

- So if R is symmetric , then $R^{-1} = R$ as $M_{R^{-1}} = M_R^T = M_R$
- M_{E_A} , M_{I_A} and M_{N_A} are both symmetric matrix

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Transitivity(传递性)

- Many relations can be chained together.
- Examples:
 - If $x = y$ and $y = z$ then $x = z$
 - If $P \subseteq Q$ and $Q \subseteq S$, then $P \subseteq S$
 - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$
- These relations are called transitive.

R on A is transitive $\Leftrightarrow$

$$(\forall a)(\forall b)(\forall c)((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Transitivity : Example

- I_A and E_A
- N_A
- $R = \{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$

Transitivity : Example

- I_A and E_A 😊
- N_A 😊
- $R = \{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$ 😞
 - $\langle 1,2 \rangle \in R, \langle 2,3 \rangle \in R$ but $\langle 1,3 \rangle \notin R$

Reflexivity, Symmetry and Transitivity

- If R_1, R_2 are reflexive, then $R_1^{-1}, R_1 \cup R_2, R_1 \cap R_2$ are also reflexive
- If R_1, R_2 are symmetric, then $R_1^{-1}, R_1 \cup R_2, R_1 \cap R_2$ are also symmetric
- If R_1, R_2 are transitive, then $R_1^{-1}, R_1 \cap R_2$ are also transitive

Reflexivity, Symmetry and Transitivity

- If R_1, R_2 are symmetric, then $R_1 \cup R_2$ are also symmetric.

Proof:

$$\langle x, y \rangle \in R_1 \cup R_2$$

$$\Leftrightarrow \langle x, y \rangle \in R_1 \vee \langle x, y \rangle \in R_2$$

$$\Leftrightarrow \langle y, x \rangle \in R_1 \vee \langle y, x \rangle \in R_2$$

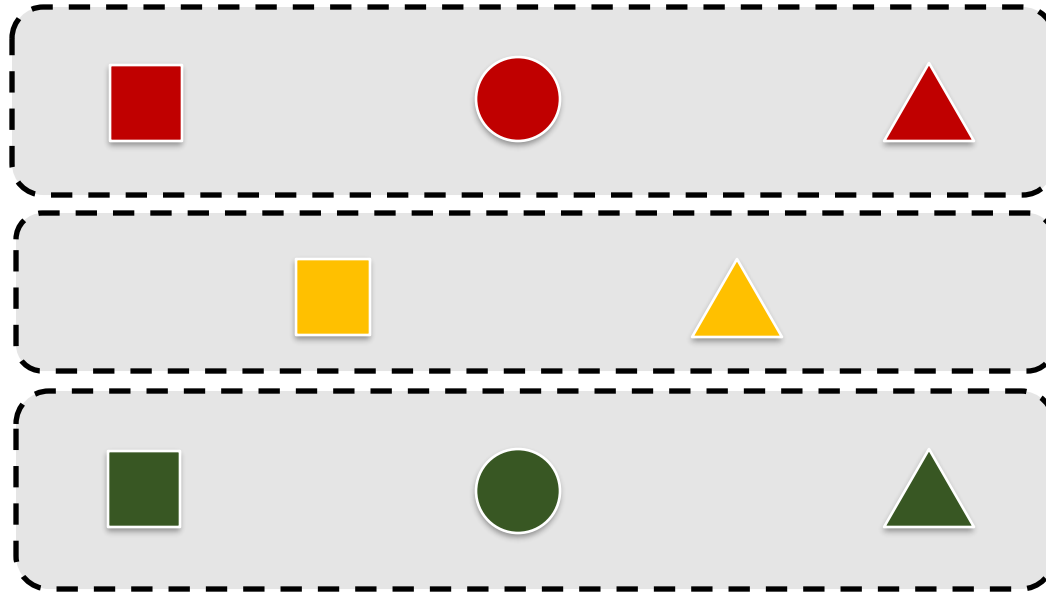
$$\Leftrightarrow \langle y, x \rangle \in R_1 \cup R_2$$

Q.E.D

Equivalence Relations(等价关系)

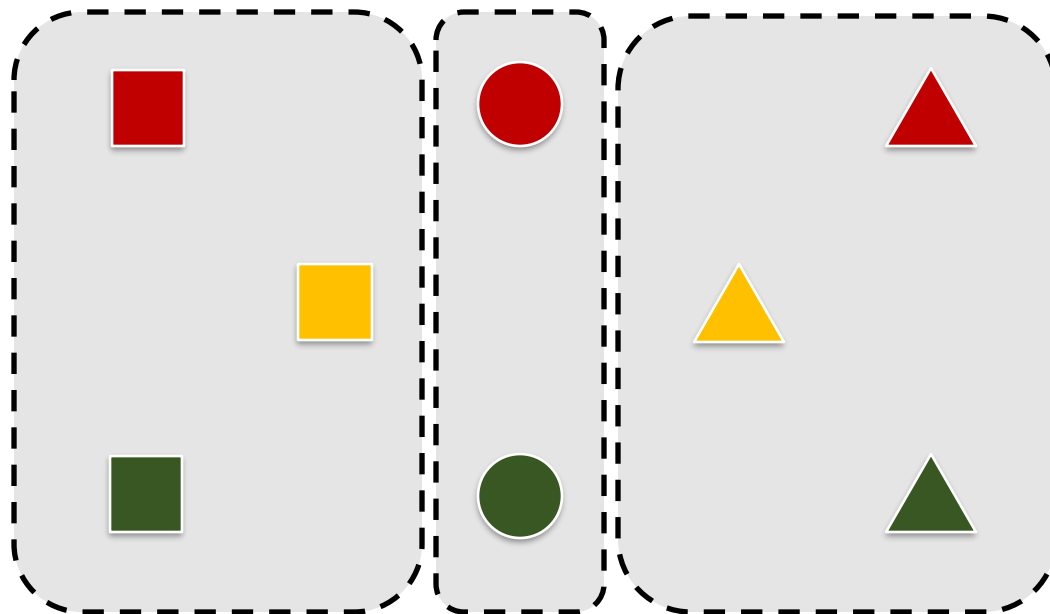
- An **equivalence relation** is a relation that is **reflexive, symmetric and transitive**.
- Some examples:
 - $x = y$
 - $x \equiv_k y$
 - x has the same color with y
 - x has the same shape with y

Partitions



aRb if a has the *color* with b

Partitions



aRb if a has the *shape* with b

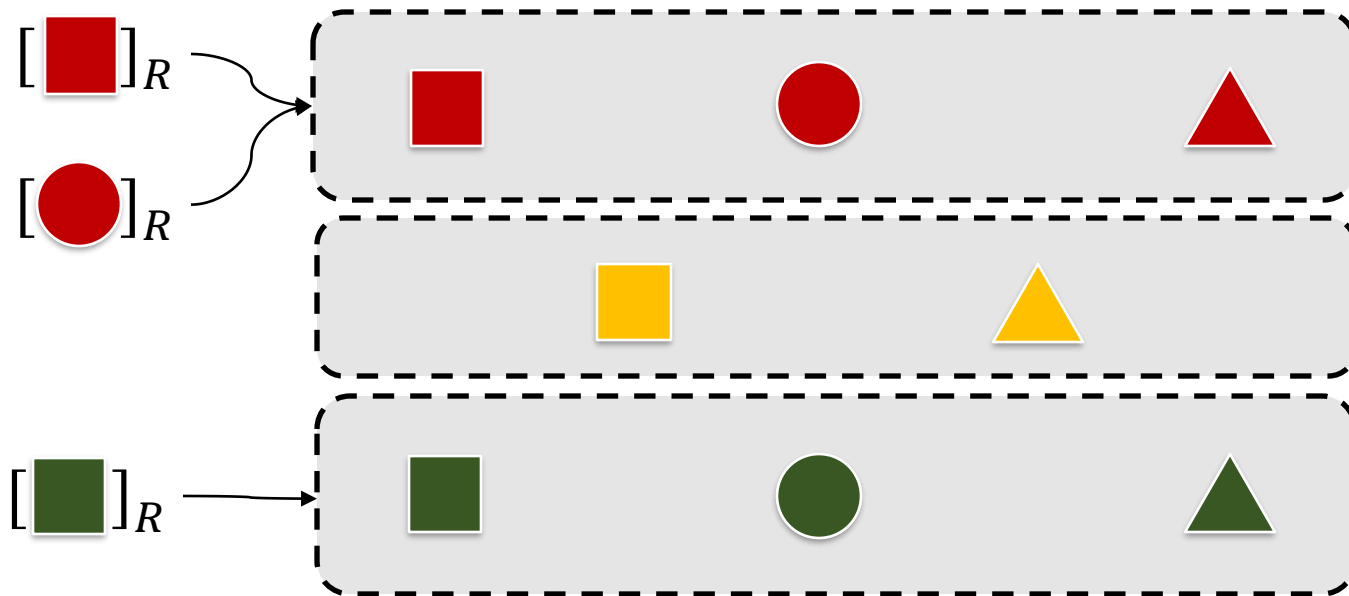
Equivalence Classes(等价类)

- Given an equivalence relation R over a set A , for any $x \in A$, the **equivalence class of x** is the set:

$$[x]_R = \{y | y \in A \wedge xRy\}$$

- Intuitively, the set $[x]_R$ contains all elements of A that are related to x by relation R .

Partitions



aRb if a has the *color* with b

Quotient Set(商集)

- Given an equivalence relation R over a set A , the quotient set of A is set of all the disjoint equivalence classes of A , denoted by A/R :

$$A/R = \{y | (\exists x)(x \in A \wedge y = [x]_R)\}$$

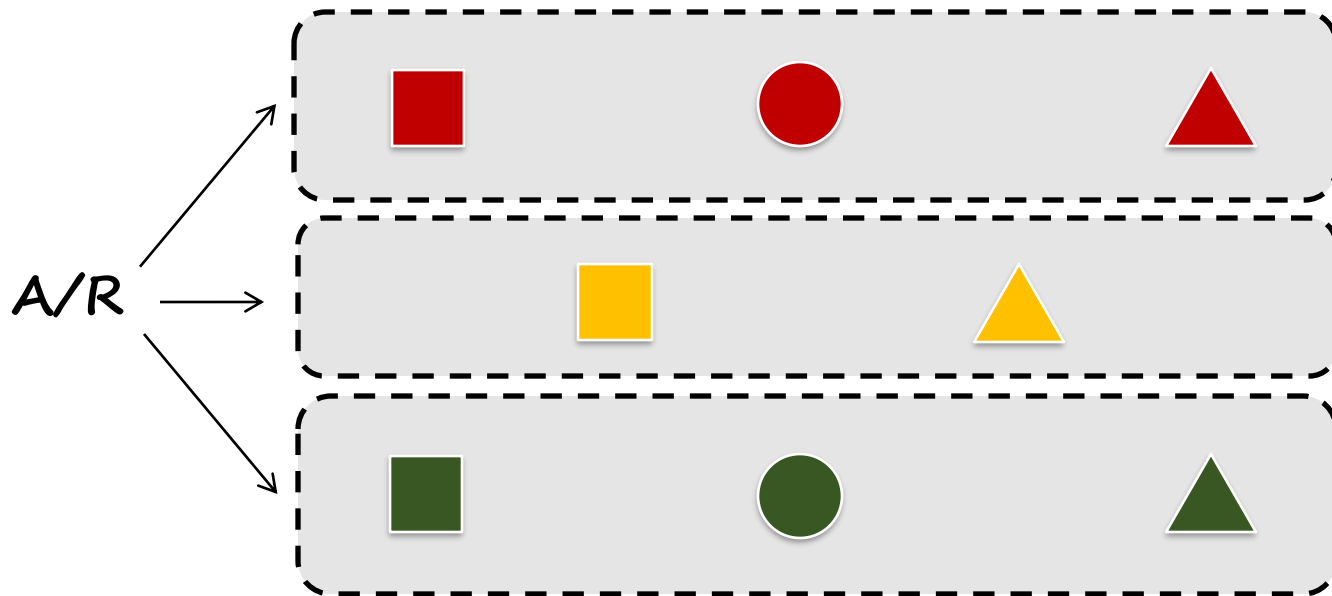
Quotient Set(商集)

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$$A/R = \{y | (\exists x)(x \in A \wedge y = [x]_R)\}$$

- For $R = \{\langle x, y \rangle | x \equiv_3 y\}$ on set $A = \{1, 2, 3 \dots 8\}$
 - $A/R = \{[1]_R, [2]_R, [3]_R\} = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6\}\}$

Partitions



aRb if a has the **color** with b

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

Definition of Partition

$$\emptyset \notin A/R$$

$$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$$

$$\cup A/R = A$$

$$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$$

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A, [x]_R, [y]_R \in A/R$:

$[x]_R \neq \emptyset$ because $x \in [x]_R$ →

Definition of Partition

$\emptyset \notin A/R$

$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

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$[x]_R \subseteq A$

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$\emptyset \notin A/R$

$[x]_R \subseteq A$

$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

$x \in [x]_R$

$\bigcup A/R = A$

$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A, [x]_R, [y]_R \in A/R$:

Definition of Partition

$[x]_R \neq \emptyset$ cause $x \in [x]_R$	$\longrightarrow$	$\emptyset \notin A/R$
$[x]_R \subseteq A$	$\longrightarrow$	$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$
$x \in [x]_R$	$\longrightarrow$	$\cup A/R = A$
$[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \emptyset$	$\longrightarrow$	$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$

Equivalence Relation and Partition

- For a partition π of set A , we can define a relation R such that aRb if a and b are in a same partition.

$$R = \{\langle a, b \rangle \mid \pi_0 \in \pi \wedge a \in \pi_0 \wedge b \in \pi_0\}$$

A partition can generate an equivalence relation.

- Given an equivalence relation R over a set A , then A/R is a partition of A

An equivalence relation can generate a partition.

*Equivalence relations precisely capture what it means
to be a partition.*

Binary relations give us a *common language* to describe *common structures*.

Equivalence Relation In Real Life

- Most modern programming languages include some sort of hash table data structure.
 - **Java: HashMap**
 - **C++: `std::unordered_map`**
 - **Python: dict**
- If you insert a key/value pair and then try to look up a key, the implementation has to be able to tell whether two keys are equal.
- Although each language has a different mechanism for specifying this, many languages describe them in similar ways...

Equivalence Relation In Real Life

- “The equals method implements an equivalence relation on non-null object references:
 - It is reflexive: for any non-null reference value `x`, `x.equals(x)` should return `true`.
 - It is symmetric: for any non-null reference values `x` and `y`, `x.equals(y)` should return `true` if and only if `y.equals(x)` returns `true`.
 - It is transitive: for any non-null reference values `x`, `y`, and `z`, if `x.equals(y)` returns `true` and `y.equals(z)` returns `true`, then `x.equals(z)` should return `true`.”

Java 8 Documentation

Equivalence Relation In Real Life

- “The equals method implements **an equivalence relation on non-null object references**:
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 - **It is transitive**: for any non-null reference values x , y , and z , if $x.equals(y)$ returns true and $y.equals(z)$ returns true, then $x.equals(z)$ should return true.”

Java 8 Documentation

Equivalence Relation In Real Life

“Each unordered associative container is parameterized by *Key*, by a function object type *Hash* that meets the Hash requirements $\langle 17.6.3.4 \rangle$ and acts as a hash function for argument values of type *Key*, and by a binary predicate *Pred* that induces an equivalence relation on values of type *Key*. Additionally, `unordered_map` and `unordered_multimap` associate an arbitrary mapped type *T* with the *Key*.”

C++14 ISO Spec, §23.2.5/3

Equivalence Relation In Real Life

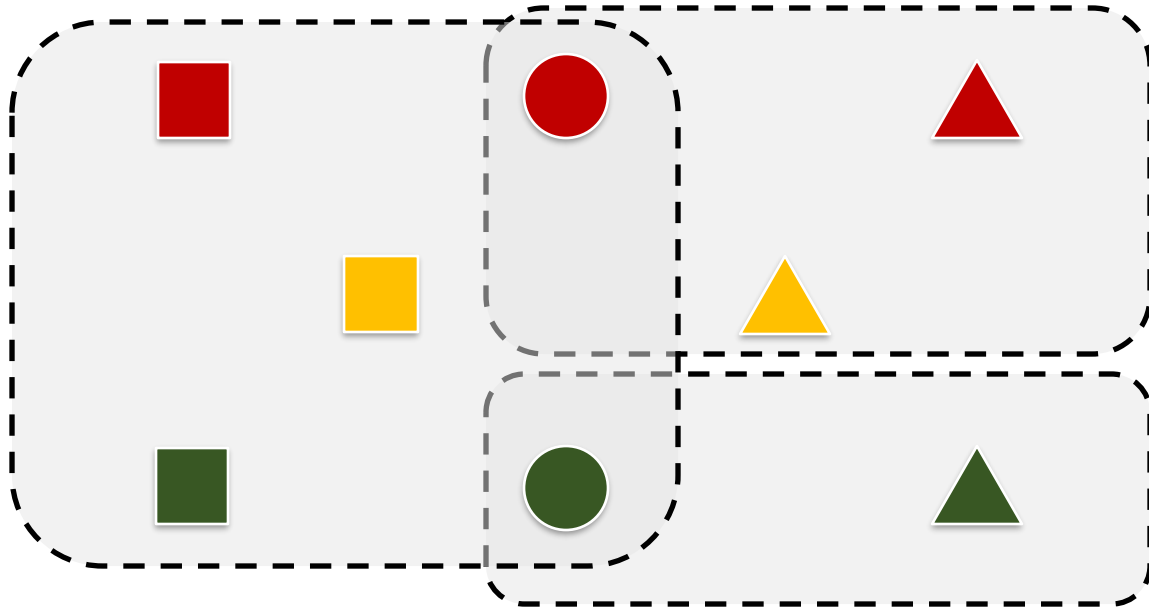
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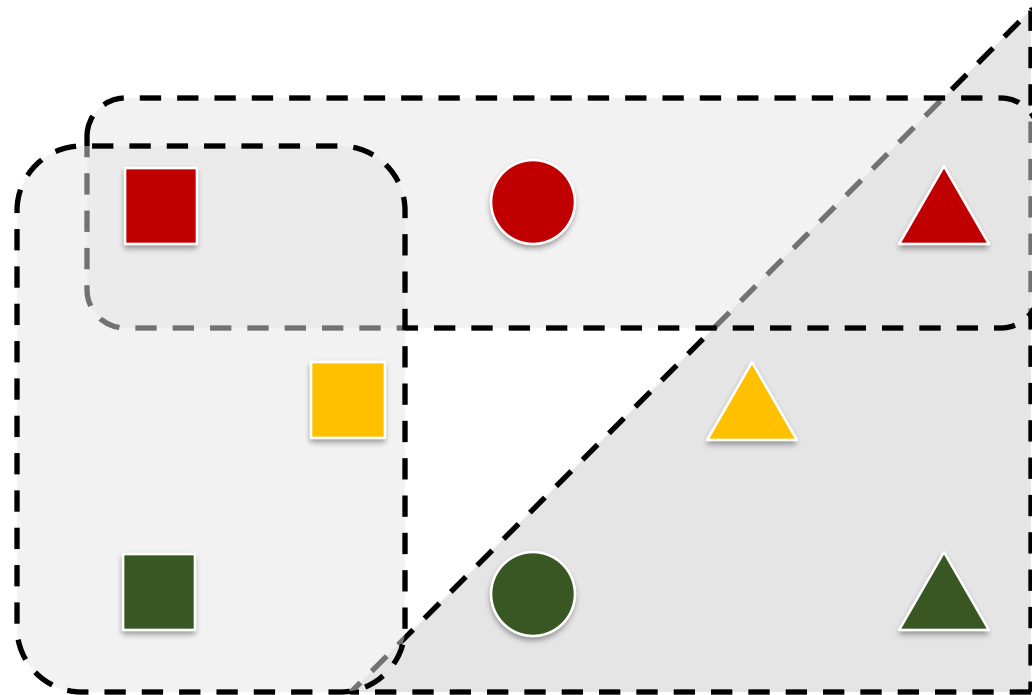
Covers (覆盖)



Covers



Covers



Covers

- A **cover of a set** is a way of splitting the set into **nonempty subsets** so that every element belongs to at least one subset.
- Intuitively, a cover contains pieces that cover every element of a set.
- **A partition is a cover, but a cover is not a partition.**

Covers

- More formally, for a non-empty set A , a set $\mathcal{C}$ is called the cover of A if:
 - $\emptyset \notin \mathcal{C}$
 - $(\forall x)(x \in \mathcal{C} \rightarrow x \subseteq A)$
 - $\bigcup \mathcal{C} = A$
 - ~~$(\forall x)(\forall y)((x \in \mathcal{C} \wedge y \in \mathcal{C} \wedge x \neq y) \rightarrow x \cap y = \emptyset)$~~

Covers

- For set $A = \{a, b, c, d\}$
 - $C_1 = \{\{a\}, \{b, c\}, \{d\}\}$
 - $C_2 = \{\{a, b, c, d\}\}$
 - $C_3 = \{\{a, b\}, \{c\}, \{a, d\}\}$
 - $C_4 = \{\{a, b, d\}\}$
- } valid covers
- } not a cover

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

$$R = \{\langle a, b \rangle \mid (\exists C_0)(C_0 \in \mathcal{C} \wedge a \in C_0 \wedge b \in C_0)\}$$

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

$$R = \{\langle a, b \rangle \mid (\exists C_0)(C_0 \in \mathcal{C} \wedge a \in C_0 \wedge b \in C_0)\}$$

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 - aRa for all $a \in A$
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Relation in Cover

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- Then we have:

- aRa for all $a \in A$

An element is in the same block with itself.

- If aRb , then bRa

If a and b are in the same block, then b and a are in the same block too.

- ~~If aRb and bRc , then aRc~~

If a and b are in the same block C_0 , b and c are also in the same block $C_1 \neq C_0$, a and b are not in the same block.

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

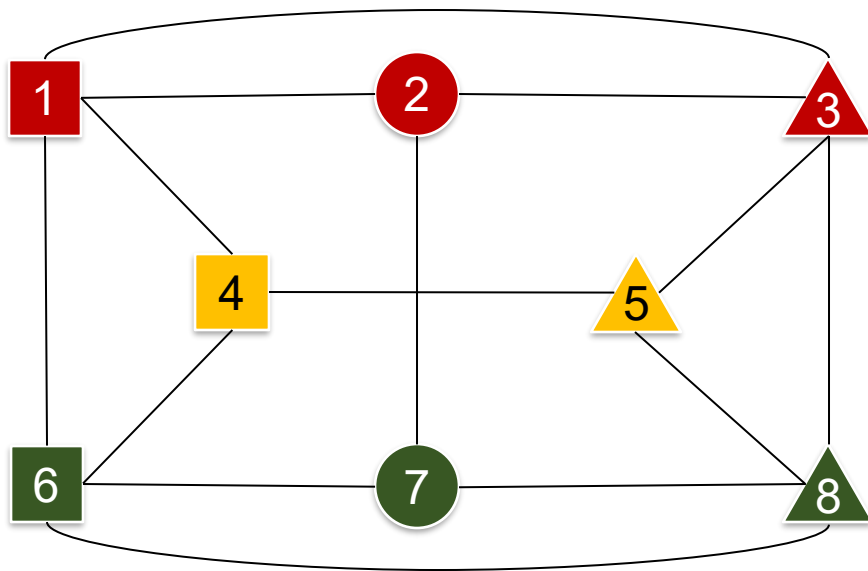
$$R = \{\langle a, b \rangle \mid (\exists C_0)(C_0 \in \mathcal{C} \wedge a \in C_0 \wedge b \in C_0)\}$$

- Then we have:
 - aRa for all $a \in A$ — Reflexivity
 - If aRb , then bRa — Symmetry
 - ~~If aRb and bRc , then aRc~~

Compatible Relations (相容关系)

- An **compatible relation** is a relation that is **reflexive and symmetric**.
- Some examples:
 - x has at least one same letter with y
 - $x \cap y \neq \emptyset$
 - x has same color or same shape with y

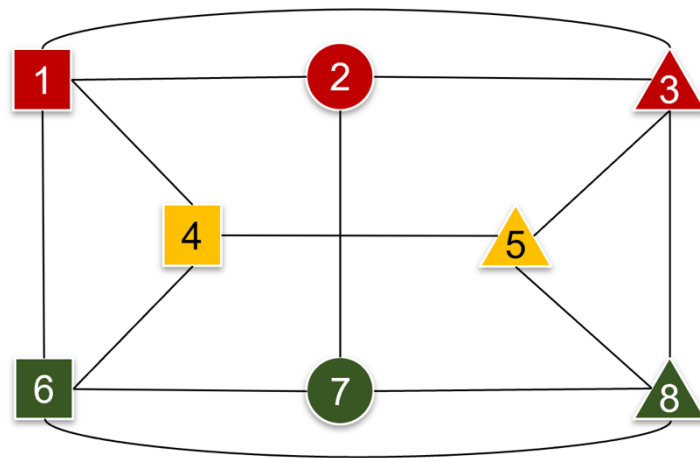
Compatible Relation



xRy if x has same *color* or same *shape* with y

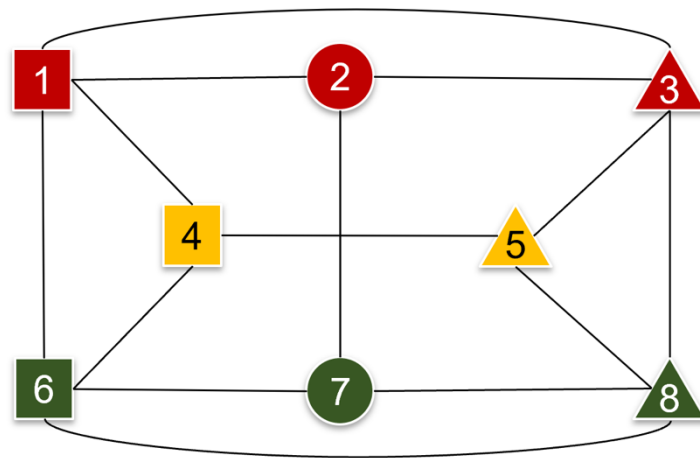
Compatible Class (相容类)

- Given a compatible relation A , C is called **compatible class** if:
 - $C \subseteq A$
 - $(x \in A) \wedge (\forall y)((y \in C) \rightarrow xRy)$
- Examples:
 - $\{1,4,6\}$
 - $\{1,4\}$
 - $\{3,8\}$
 - ...



Maximal Compatible Class(极大相容类)

- A compatible class C is called **maximal compatible class** if C is not the subset of any compatible class, denote by C_R .
- Examples:
 - $\{1,4,6\}$
 - $\{1,2,3\}$
 - $\{2,7\}$
 - $\{3,5,8\}$
 - ...

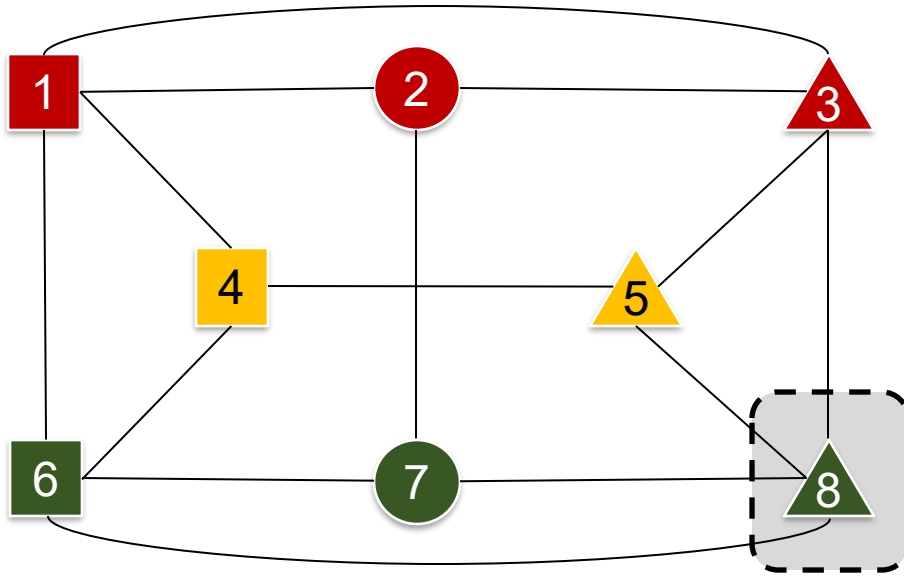


Maximal Compatible Class

For any compatible class C on set A , there exists a maximal compatible class C_R such that $C \subseteq C_R$

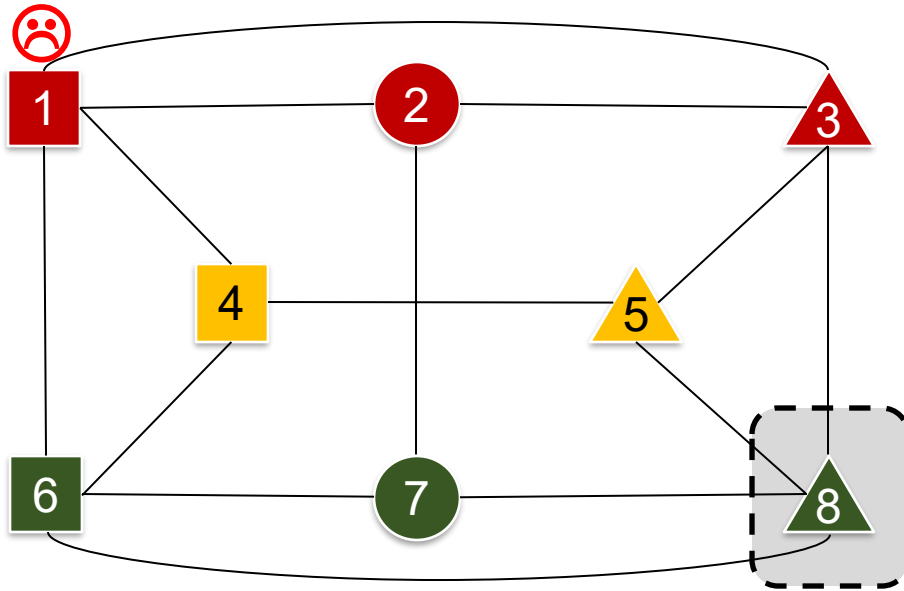
- We can generate a sequence such that $C_0 \subseteq C_1 \subseteq C_2 \dots \subseteq C_R$
 - $C_0 = C$
 - $C_{i+1} = C_i \cup \{a_j\}$, where j is the minimal index such that $a_j \notin C_i \wedge (x \in C_i \rightarrow a_j R x)$

Maximal Compatible Class



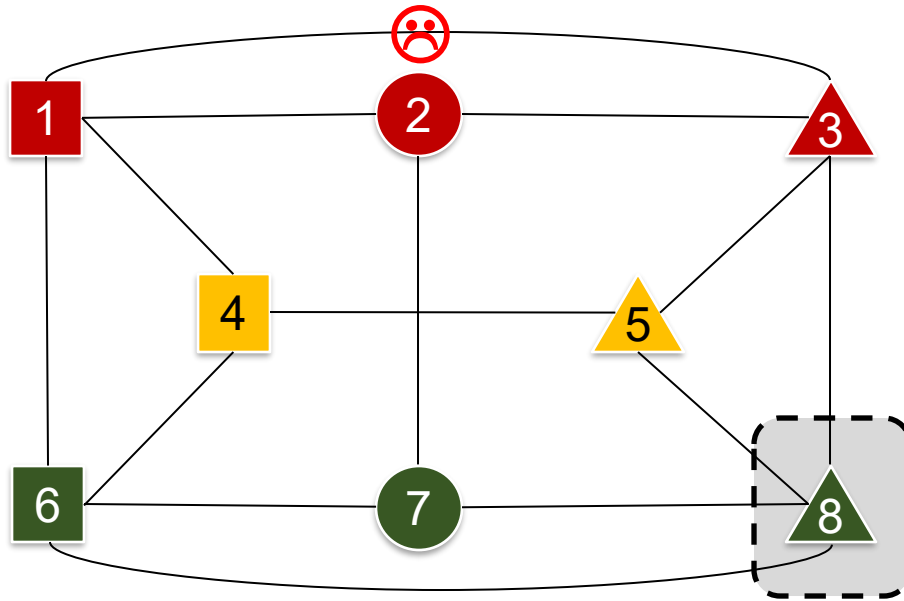
$$C_0 = C = \{8\}$$

Maximal Compatible Class



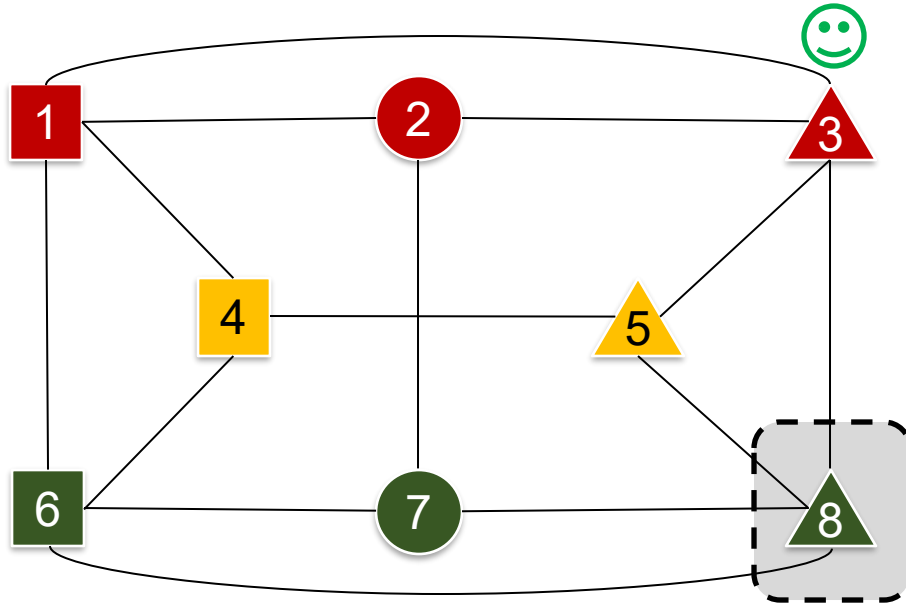
$$C_0 = C = \{8\}$$

Maximal Compatible Class



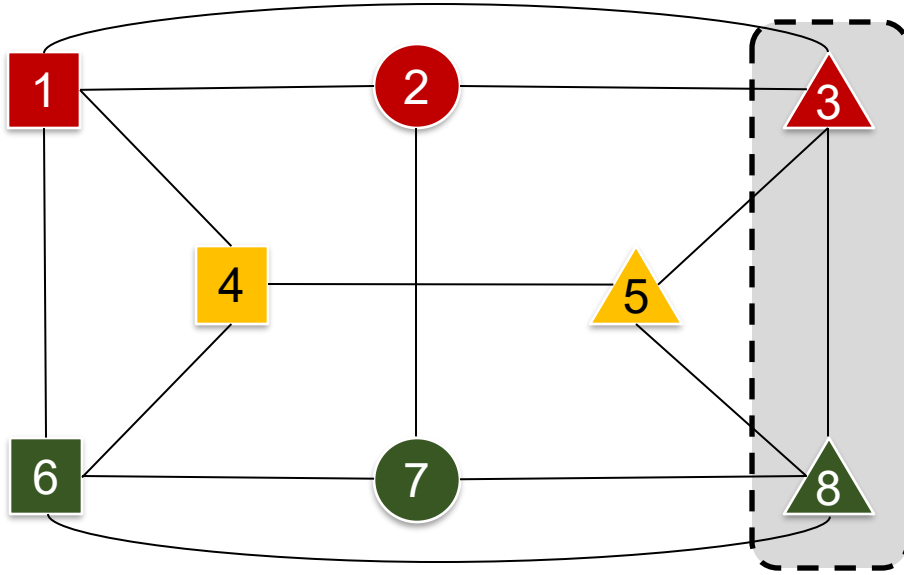
$$C_0 = C = \{8\}$$

Maximal Compatible Class



$$C_0 = C = \{8\}$$

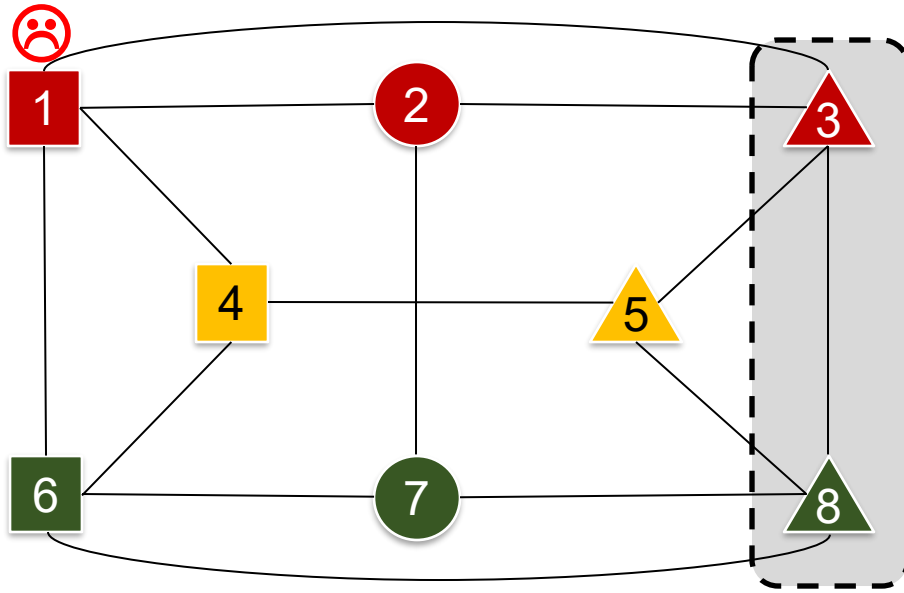
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

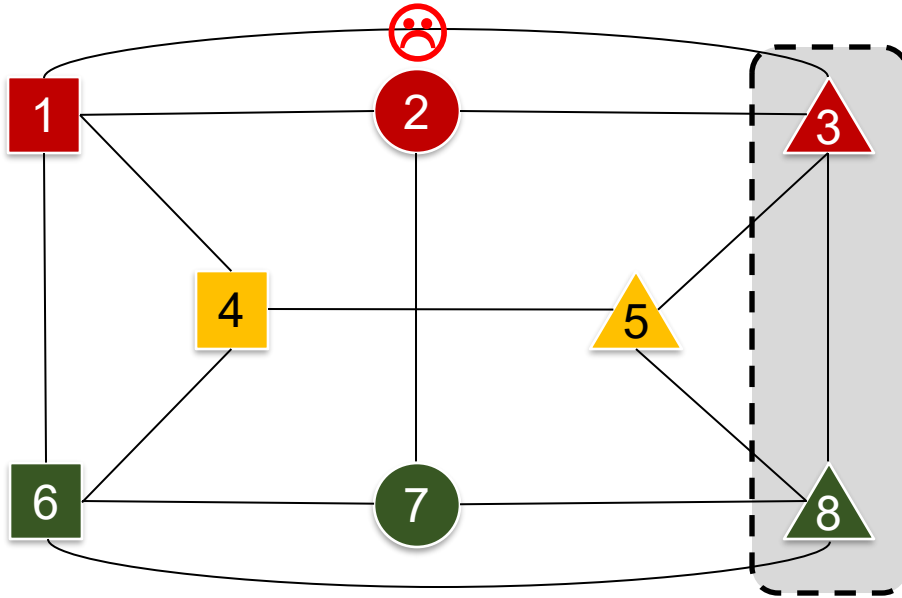
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

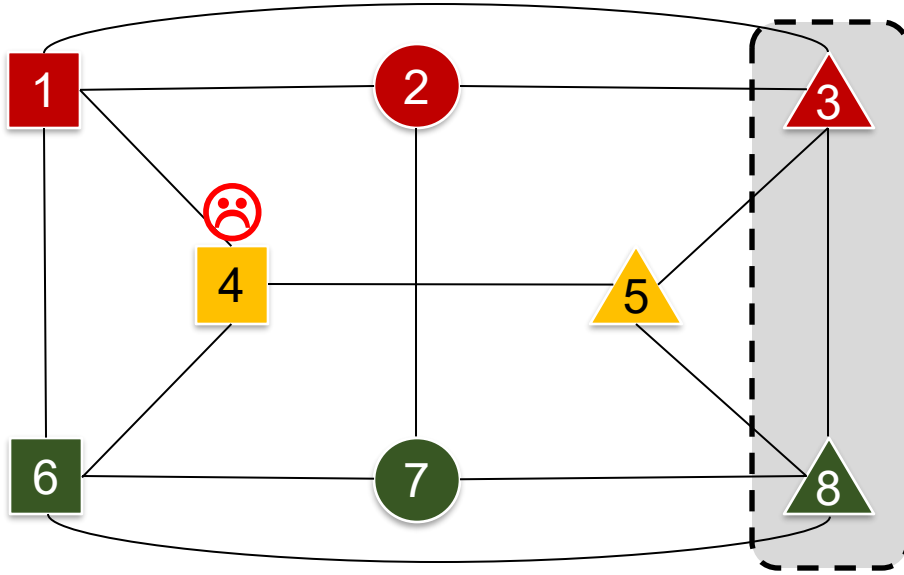
Maximal Compatible Class



$$C_0 = C = \{8\}$$

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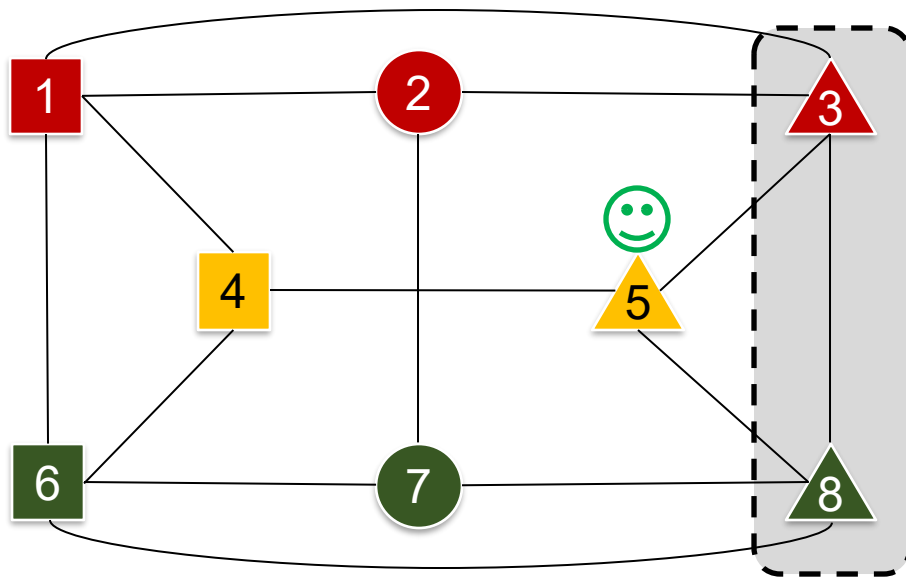
Maximal Compatible Class



$$C_0 = C = \{8\}$$

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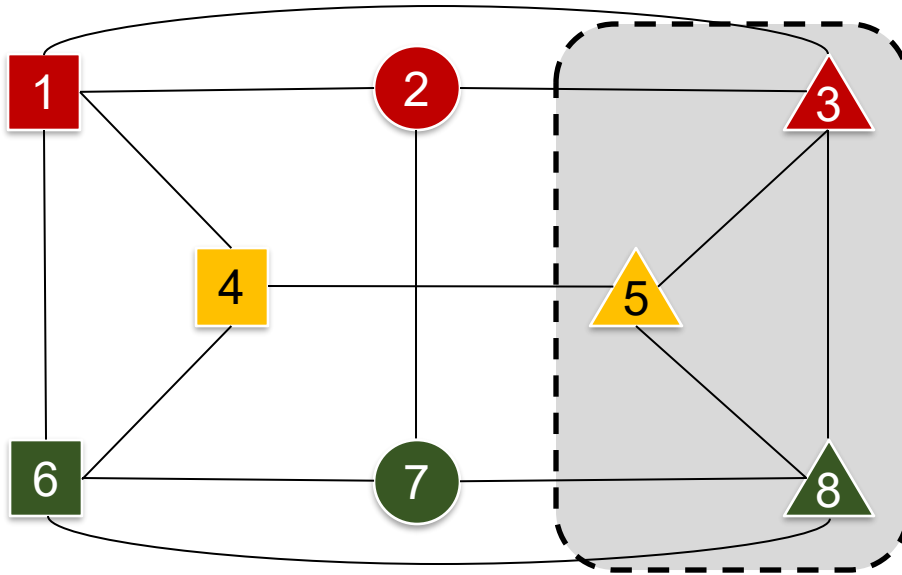
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

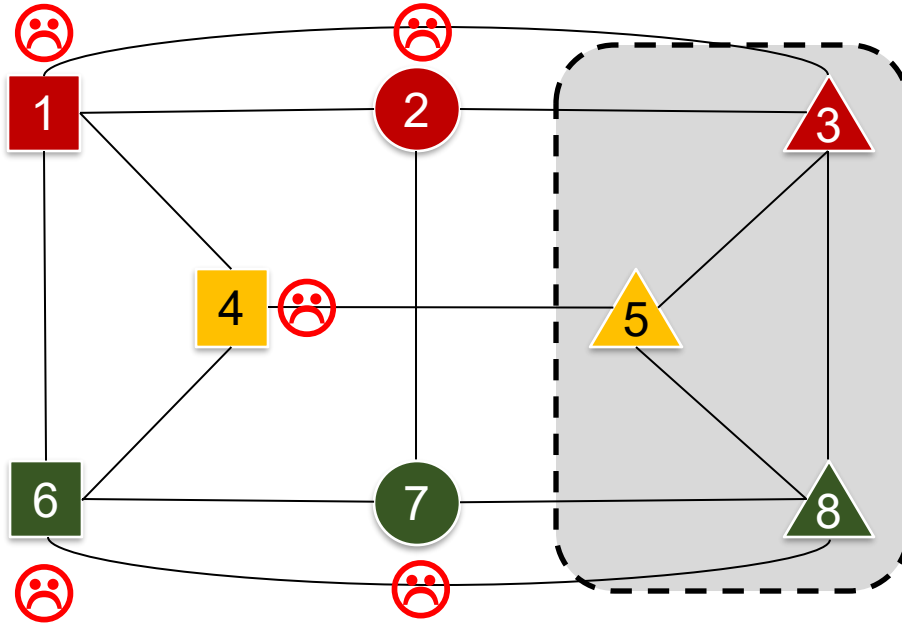
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

$$C_2 = \{3, 5, 8\}$$



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

$$C_2 = \{3, 5, 8\}$$

$$C_R = C_2 = \{3, 5, 8\}$$

Compatible Relation and Covers

- Given a compatible relation R on a set A , the set of all maximal compatible classes is a cover of A , called the complete cover (完全覆盖) of A , denote by $C_R(A)$.
 - $C_R(A)$ is unique.
- A compatible relation of A can generate a complete cover of A
- A cover of A can generate a compatible relation of A
- Two different covers can generate the same compatible relation.



Next

Capture the structure of a problem ⟨continued⟩

- Partial Order, total order.
- Closures

Other fundamental tools.

- Functions